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AN INEQUALITY BETWEEN THE p- AND $(p,1)$-SUMMING NORM OF FINITE RANK OPERATORS FROM $C(K)$-SPACES

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ABSTRACT

We show, for any operator T from a $C(K)$-space into a Banach space with $\operatorname{rank}(T) \le n$, the inequality

$$\pi_p(T) \le C(1+\log n)^{1-1/p}\pi_{p,1}(T), \qquad 1<p<\infty,$$

where $C \le 4.671$ is a numerical constant. The factor $(1+\log n)^{1-1/p}$ is asymptotically correct. This inequality extends a result of Jameson to $p \ne 2$. Several applications are given — one is a positive solution of a conjecture of Rosenthal and Szarek: For $1 \le p < q < 2$,

$$\sup_{\substack{\|T: l_q^n \to l_p^n\| \le 1 \\ \|S: l_q^n \to l_p^n\| \le 1}} \|T \otimes S : l_q^n \otimes_q l_q^n \to l_p^n \otimes_p l_p^n\| \asymp (1+\log n)^{1/q}.$$

1. Introduction

We recall some basic definitions. The Banach space of all (bounded linear) operators from a Banach space E into a Banach space F is denoted by $\mathcal{L}(E,F)$. We say an operator $T \in \mathcal{L}(E,F)$ is (r,p)-summing, $1 \le p \le r \le \infty$, if there is a constant $\sigma \ge 0$ such that

$$\left(\sum_{i=1}^m \|Tx_i\|^r\right)^{1/r} \le \sigma \sup_{\|a\|\le 1}\left(\sum_{i=1}^m |\langle x_i, a\rangle|^p\right)^{1/p}$$

for all finite families $x_1,\dots,x_m \in E$. Put

$$\pi_{r,p}(T) := \inf \sigma,$$

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then the class of all (r,p)-summing operators forms a Banach ideal $[\mathcal{P}_{r,p}, \pi_{r,p}]$. For $r = p$ we have the Banach ideal $[\mathcal{P}_p, \pi_p]$ of p-summing operators. Furthermore, in this paper $C(K)$ denotes the Banach space of continuous functions over a compact Hausdorff space. Moreover, $L_{p,q}(\mu)$ and $l_{p,q}$ denote the well-known Lorentz function and sequence spaces, respectively, with the abbreviations $L_p := L_{p,p}$ and $l_p := l_{p,p}$ for $q = p$. Pisier's factorization theorem of $(p,1)$-summing operators on $C(K)$-spaces states that such an operator can be factored through the embedding operator from $C(K)$ into the Lorentz space $L_{p,1}(K,\mu)$, for some probability measure μ on K. This result corresponds to the Grothendieck–Pietsch factorization theorem for p-summing operators. A result of Maurey (Lemma A in section 2) states that each $(p,1)$-summing operator from a $C(K)$-space into a Banach space is r-summing for $r > p$. This result may also be checked by Pisier's factorization theorem. It is natural to ask what is the ratio in the limit case $r = p$ of the p- and $(p,1)$-summing norm of finite rank operators from $C(K)$-spaces. The answer is given by the inequality stated in the abstract (Theorem 1 in section 2). This inequality has useful applications. So we prove a precise version of a result of Saphar [Sa] for finite rank operators from $L_1(\mu)$ into $L_p(\nu)$, $2 < p < \infty$, by using Theorem 1 and a result of Schechtman [S]. Furthermore, we provide asymptotically optimal estimates between (q,p)-mixing and (r,p)-summing norms of finite rank operators between arbitrary Banach spaces (Proposition 3 in section 3.2). Moreover, we answer a conjecture of Rosenthal and Szarek [R–S] in the positive (Proposition 4 in section 3.3). Finally, according to the pattern of Tomczak-Jaegermann [T] we may even show the inequality

$$\pi_p(T) \le C_p(1 + \log n)^{1-1/p}\pi_{p,1}^{(n)}(T),$$

for rank n operators from $C(K)$-spaces in the case $p \ge 2$, where $\pi_{p,1}^{(n)}(T)$ stands for the $(p,1)$-summing norm of operators generated by n-vectors. For more details we refer to the final remarks.

2. The main theorem

The main result of the paper is stated in the following theorem.

THEOREM 1. *Let $1 < p < \infty$ and let n be any natural number. Then for all operators $T \in \mathcal{L}(C(K),E)$ from a $C(K)$-space into a Banach space E with* $\operatorname{rank}(T) \le n$,

$$\pi_p(T) \le C(1 + \log n)^{1-1/p}\pi_{p,1}(T),$$

where $C \le 4.671$ is a numerical constant.

REMARKS. (i) The special case $p = 2$ is due to Jameson [J], but his method does not work for $p \neq 2$.

(ii) A modification of the example in Jameson [J] (cf. Montgomery-Smith [Mo]) shows that the factor $(1 + \log n)^{1-1/p}$ appearing in the above inequality is asymptotically correct for $1 < p < \infty$: Indeed,

$$\pi_p(I_n : l_\infty^n \to l_{p,1}^n) \asymp n^{1/p}(1 + \log n)^{1-1/p}$$

and

$$\pi_{p,1}(I_n : l_\infty^n \to l_{p,1}^n) \asymp n^{1/p},$$

where I_n denote the identity operators.

For the proof of Theorem 1 we need two ingredients. The first one is a result of Maurey (cf. also Pisier [P]) and the second one is implicitly contained in [C].

LEMMA A (Maurey [M1]). *Let $1 \le p < r \le \infty$ and let $T \in \mathcal{P}_{p,1}(C(K), E)$. Then*

$$\pi_r(T) \le \frac{\Gamma(1 - r'/p')^{1/r'}}{\Gamma(1/p)} \pi_{p,1}(T),$$

where Γ stands for the well-known Gamma function and r' is the conjugate index to r, $1/r' := 1 - 1/r$.

Since for operators from $C(K)$-spaces the p-summing and p-integral norms coincide (cf. [Pi], [T], [D–F]) we conclude from Proposition 3′ of [C] and the injectivity of the p-summing norm the following lemma.

LEMMA B ([C]). *Let $1 \le p \le r \le \infty$ and let $T \in \mathcal{L}(C(K), E)$ be an operator with* $\operatorname{rank}(T) \le n$. *Then*

$$\pi_p(T) \le n^{\max(r'/2,1)(1/p - 1/r)} \pi_r(T).$$

PROOF OF THEOREM 1. Given $T \in \mathcal{L}(C(K), E)$ with $\operatorname{rank}(T) \le n$. Combining Lemmas A and B we get

$$\pi_p(T) \le \frac{\Gamma(1 - r'/p')^{1/r'}}{\Gamma(1/p)} n^{\max(r'/2,1)(1/p - 1/r)} \pi_{p,1}(T) \qquad \text{for } 1 < p < r \le \infty.$$

Note that

$$\Gamma\left(1 - \frac{r'}{p'}\right)^{1/r'} = \left(1 - \frac{r'}{p'}\right)^{-1/r'} \Gamma\left(2 - \frac{r'}{p'}\right)^{-1/r'} \le \left(1 - \frac{r'}{p'}\right)^{-1/r'} \le \left(\frac{1}{p} - \frac{1}{r}\right)^{-1/r'}$$

for $1 \le r' < p' < \infty$ and

$$\Gamma\left(\frac{1}{p}\right) = p\Gamma\left(1+\frac{1}{p}\right) \ge p \inf_{1\le p<\infty} \Gamma\left(1+\frac{1}{p}\right) = \frac{p}{a},$$

where $a = 1.129\ldots$. Hence

$$\pi_p(T) \le f_{p,n}(r)\pi_{p,1}(T)$$

with

$$f_{p,n}(r) = \frac{a}{p}\left(\frac{1}{p}-\frac{1}{r}\right)^{-1/r'} n^{\max(r'/2,1)(1/p-1/r)}.$$

Choose r such that

$$\frac{1}{p}-\frac{1}{r} = \frac{1}{r'}-\frac{1}{p'} = \frac{1}{\max(p,p')}\cdot(1+\log n)^{-1}.$$

Then

$$\frac{1}{p'} < \frac{1}{r'} \le \frac{1}{p'} + \frac{1}{\max(p,p')} \le 1,$$

so that

$$\begin{aligned} f_{p,n}(r) &= \frac{a}{p}(\max(p,p'))^{1/r'}(1+\log n)^{1/r'} n^{[\max(r'/2,1)/\max(p,p')]\cdot(1+\log n)^{-1}} \\ &\le \frac{a}{p}(\max(p,p'))^{1/p'+1/\max(p,p')} e^{1/e\max(p,p')} e^{\max(p'/2,1)/\max(p,p')} \\ &\quad \times (1+\log n)^{1/p'}. \end{aligned}$$

Therefore, the estimates

$$\frac{1}{p}\max(p,p')^{1/p'+1/\max(p,p')} \le \frac{1}{p}\max(p,p')^{1/p'}e^{1/e} \le \max(1,e^{1/e})e^{1/e} \le e^{2/e},$$

$$e^{\max(p'/2,1)/\max(p,p')} \le e^{1/2} \quad \text{and} \quad e^{1/e\max(p,p')} \le e^{1/2e}$$

imply the desired inequality

$$\pi_p(T) \le C(1+\log n)^{1-1/p}\pi_{p,1}(T),$$

where $C \le a\cdot e^{5/2e+1/2} \le 4.671$. ∎

3. Applications

We now deal with several applications of Theorem 1.

3.1. On a result of Saphar

A result of Saphar [Sa] states that each operator $T \in \mathcal{L}(L_1(\mu), L_p(\nu))$, $2 < p < \infty$, is q-summing for $q > p$. In this section we consider the limit case $q = p$. For doing this we recall r-stable random variables, $1 < r < 2$. A random variable f is said to be r-stable if its Fourier transform is equal to $e^{-|t|^r}$. By (f_k) we denote a sequence of independent r-stable random variables defined on the interval $[0,1]$ with the Lebesgue measure. It is well known that

$$\left(\int_0^1 \left|\sum_{k=1}^m b_k f_k(t)\right|^s dt\right)^{1/s} = a_{r,s}\left(\sum_{k=1}^m |b_k|^r\right)^{1/r} \qquad \text{for } 1 \le s < r < 2,$$

where $a_{r,s}$ is a constant depending on r and s. This equation means that l_r^m can be isometrically embedded into $L_s[0,1]$ for $1 \le s < r < 2$. Moreover,

$$\left(\int_0 \sum_{k=1}^m |f_k(t)|^r\right)^{1/r} dt \ge c(1 + \log n)^{1/r} n^{1/r},$$

where $c > 0$ is a positive constant (cf. [M–P],[T]).

Proposition 2. (i) *Let $2 < p < \infty$ and let n be any natural number. Then for all operators $T \in \mathcal{L}(L_1(\mu), L_p(\nu))$ with* $\operatorname{rank}(T) \le n$,

$$\pi_p(T) \le C_p(1 + \log n)^{1-1/p}\|T\|,$$

where C_p is a constant depending only on p.

(ii) *For $2 < p < \infty$ there exists a sequence of operators*

$$T_n \in \mathcal{L}(L_1[0,1], l_p) \textit{ with } \operatorname{rank}(T_n) \le n$$

such that

$$\pi_p(T_n) \ge C_p(1 + \log n)^{1-1/p} \quad \textit{and} \quad \|T_n\| \le 1,$$

where $C_p > 0$ is a constant depending only on p.

Proof. (i) First we show for operators $T \in \mathcal{L}(L_1(\mu), l_p^m)$,

$$\pi_p(T) \le 2C(1 + \log m)^{1-1/p}\|T\|,$$

where C is the constant of Theorem 1. Indeed, the p'-stable random variables provide a metric injection J from $l_{p'}^m$ into $L_1[0,1]$. Thus $Q := J'$ defines a metric surjection from $L_\infty[0,1]$ onto l_p^m. By the metric lifting property of $L_1(\mu)$ we may factorize each operator T as follows:

$$\begin{array}{ccc} & & L_{\infty[0,1]} \\ & \overset{\hat{T}}{\nearrow} & \big\downarrow Q \\ L_1(\mu) & \xrightarrow{\ T\ } & l_p^m \end{array} \quad ,$$

where $\hat{T}$ is an operator with $\|\hat{T}\| \le 2\|T\|$. Applying Theorem 1 we infer

$$\begin{aligned} \pi_p(T) &\le \|\hat{T}\|\,\pi_p(Q) \le 2\|T\|\,\pi_p(Q) \\ &\le 2C\|T\|\,(1+\log m)^{1-1/p}\pi_{p,1}(Q). \end{aligned}$$

Because of

$$\pi_{p,1}(Q) \le \|Q\|\,\pi_{p,1}(I_m : l_p^m \to l_p^m) \le \pi_{p,1}(I_m : l_p^m \to l_p^m)$$

and

$$\pi_{p,1}(I_m : l_p^m \to l_p^m) = 1$$

(cf. [K],[Pi],[T]) we get the desired inequality. Now let $T \in \mathcal{L}(L_1(\mu), L_p(\nu))$ with $\mathrm{rank}(T) \le n$. Put $E_n := T(L_1)$. Then $\dim E_n \le n$ and we have, for the operator $T_0 : L_1(\mu) \to E_n$ induced by T, that $\pi_p(T) = \pi_p(T_0)$ and $\|T\| = \|T_0\|$. By a result of Schechtman [S] (cf. also Bourgain/Lindenstrauss/Milman [B–L–M]) we may embed E_n 2-isomorphically into an l_p^m with $m \le K(p)n^{1+p/2}$. Hence,

$$\pi_p(T) = \pi_p(T_0) \le 2\pi_p(J_nT_0),$$

where J_n is the 2-isomorphic embedding from E_n into l_p^m. By the first step of the proof we conclude that

$$\begin{aligned} \pi_p(J_nT_0) &\le 2C(1+\log m)^{1-1/p}\|J_nT_0\| \\ &\le 2C(1+\log m)^{1-1/p}\|T\|. \end{aligned}$$

Therefore

$$\pi_p(T) \le 4C(1+\log m)^{1-1/p}\|T\|.$$

From

$$\log m \le \log K(p)n^{1+p/2} \le \max(\log K(p), p/2)(1+\log n)$$

we finally get the desired estimate

$$\pi_p(T) \le C_p(1+\log n)^{1-1/p}\|T\|,$$

where $C_p \le 4C(1+\max(\log K(p), p/2))^{1-1/p}$.

For the proof of (ii) we follow a similar idea of Kwapien (cf. [T]). Let $(f_k)_{k=1}^{\infty}$ be a sequence of independent p'-stable random variables. For each n we may find $g_1^{(n)},\dots,g_n^{(n)} \in L_\infty[0,1]$ such that

$$\sup_{t\in[0,1]} \sum_{k=1}^{n} |g_k^{(n)}(t)|^p = 1$$

and

$$\left(\int_0^1 \sum_{k=1}^{n} |f_k(t)|^{p'}\right)^{1/p'} dt = \left|\int_0^1 \sum_{k=1}^{n} f_k(t) g_k^{(n)}(t)\, dt\right|.$$

Define operators $A_n : l_{p'} \to L_1[0,1]$ and $T_n : L_1[0,1] \to l_p$ by

$$A_n x := \sum_{k=1}^{n} \langle x, e_k\rangle f_k, \qquad x \in l_{p'},$$

and

$$T_n f := \sum_{k=1}^{n} \langle f, g_k^{(n)}\rangle e_k, \qquad f \in L_1[0,1],$$

respectively. Obviously,

$$\|A_n : l_{p'} \to L_1[0,1]\| \le a_{p',1} \quad \text{and} \quad \|T_n : L_1[0,1] \to l_p\| \le 1.$$

On the other hand we get

$$\begin{aligned}
cn^{1/p'}(1+\log n)^{1/p'} &\le \int_0^1 \left(\sum_{k=1}^{n} |f_k(t)|^{p'}\right)^{1/p'} dt \\
&= \left|\int_0^1 \sum_{k=1}^{n} f_k(t) g_k^{(n)}(t)\, dt\right| = \left|\sum_{k=1}^{n} \langle f_k, g_k^{(n)}\rangle\right| \\
&\le n^{1/p'}\left(\sum_{k=1}^{n} |\langle f_k, g_k^{(n)}\rangle|^p\right)^{1/p} \le n^{1/p'}\left(\sum_{k=1}^{n}\sum_{l=1}^{n} |\langle f_k, g_l^{(n)}\rangle|^p\right)^{1/p} \\
&\le n^{1/p'}\left(\sum_{k=1}^{n} \|T_n A_n e_k\|_p^p\right)^{1/p} \le n^{1/p'}\pi_p(T_n A_n) \\
&\le n^{1/p'}\pi_p(T_n)\|A_n\| \le n^{1/p'}\pi_p(T_n) a_{p',1},
\end{aligned}$$

which yields the desired assertion (ii),

$$\pi_p(T_n) \geq \frac{c}{a_{p',1}} (1 + \log n)^{1-1/p} \quad \text{and} \quad \|T_n\| \leq 1. \qquad \blacksquare$$

3.2. Inequalities between (q,p)-mixing and (r,p)-summing norms of finite rank operators

Let $1 \leq p,q \leq \infty$. An operator $T \in \mathfrak{L}(E,F)$ from a Banach space E into a Banach space F is said to be (q,p)-mixing if for all Banach spaces G and all operators $S \in \mathfrak{P}_q(F,G)$ the composition operator ST is p-summing. Put

$$\mu_{q,p}(T) := \sup\{\pi_p(ST) : \pi_q(S) \leq 1\}.$$

Then the class $[\mathfrak{M}_{q,p}, \mu_{q,p}]$ of all (q,p)-mixing operators is a Banach ideal (cf. [Pi], [D–F]). We now recall some properties about (q,p)-mixing operators, most of which can be found at least implicitly in Maurey's thesis [M2]. The theory of these operators appeared in a condensed form in Pietsch [Pi] and Puhl [Pu] (see also [D–F]).

By definition we obviously have

$$[\mathfrak{M}_{q,p}, \mu_{q,p}] = [\mathfrak{L}, \|\cdot\|] \qquad \text{for } 1 \leq q \leq p \leq \infty,$$

and

$$[\mathfrak{M}_{\infty,p}, \mu_{\infty,p}] = [\mathfrak{P}_p, \pi_p], \qquad 1 \leq p \leq \infty.$$

Furthermore, if $1/r + 1/q = 1/p$, then

$$[\mathfrak{P}_{r,q}, \pi_{r,q}] \cdot [\mathfrak{M}_{q,p}, \mu_{q,p}] \subseteq [\mathfrak{P}_{s,p}, \pi_{s,p}]$$

for $1 \leq p \leq q \leq \infty$ and $1/p - 1/s = 1/q - 1/r$. The special case $r = q$, $s = p$ comes out from the definition. For our purposes we also need the following characterization: An operator $T \in \mathfrak{L}(E,F)$ is (q,p)-mixing iff for all $C(K)$-spaces, K a compact Hausdorff space, all operators $S \in \mathfrak{P}_{p'}(C(K),E)$ the composition operator TS is q'-summing, and in this case

$$\mu_{q,p}(T) = \sup\{\pi_{q'}(TS) : \pi_{p'}(S) \leq 1\}.$$

For $p = 1$ we have

$$[\mathfrak{M}_{q,1}(C(K),E), \mu_{q,1}] = [\mathfrak{P}_{q'}(C(K),E), \pi_{q'}].$$

The next result (again due to Maurey [M2]) shows that the (q,p)-mixing and (r,p)-summing operators are closely related: Namely, if $1/r + 1/q = 1/p$, then

$$[\mathfrak{M}_{q,p}, \mu_{q,p}] \subseteq [\mathcal{P}_{r,p}, \pi_{r,p}] \quad \text{and} \quad \mathcal{P}_{r,p} \subseteq \mathfrak{M}_{q-\epsilon,p} \quad \text{for } \epsilon > 0.$$

It is natural to ask: what is the ratio between the (q,p)-mixing and (r,p)-summing norm of finite rank operators in the "border case" $\epsilon = 0$ and $1/r + 1/q = 1/p$? We prove the following statement, which is useful for investigating norms of tensor product operators.

PROPOSITION 3. *Let $1 \leq p \leq q \leq \infty$ and $1/r + 1/q = 1/p$. Then for any natural number n and any operator $T \in \mathcal{L}(E,F)$ between arbitrary Banach spaces E and F with* $\operatorname{rank}(T) \leq n$,

$$\mu_{q,p}(T) \leq C(1 + \log n)^{1/q} \pi_{r,p}(T),$$

where C is the numerical constant from Theorem 1.

PROOF. Let $S \in \mathcal{P}_{p'}(C(K),E)$ be an arbitrary p'-summing operator from a $C(K)$-space into E. By Theorem 1 we have

$$\pi_{q'}(TS) \leq C(1 + \log n)^{1/q} \pi_{q',1}(TS).$$

Using the above product formula

$$\pi_{q',1}(TS) \leq \pi_{r,p}(T)\mu_{p,1}(S), \qquad 1/r + 1/q = 1/p$$

and

$$\mu_{p,1}(S) \leq \pi_{p'}(S)$$

we check

$$\pi_{q'}(TS) \leq C(1 + \log n)^{1/q} \pi_{r,p}(T)\pi_{p'}(S).$$

Thus, the above characterization of (q,p)-mixing operators yields

$$\mu_{q,p}(T) = \sup\{\pi_{q'}(TS) : \pi_{p'}(S) \leq 1\} \leq C(1 + \log n)^{1/q} \pi_{r,p}(T)$$

for $1/r + 1/q = 1/p$. ∎

REMARK. The factor $(1 + \log n)^{1/q}$ in the inequality of Proposition 3 is asymptotically correct at least in the cases $1 = p < q < \infty$ and $1 < p < q < 2$, respectively. Indeed, if $1 = p < q < \infty$, then the inequality of Proposition 3 reduces to the inequality of Theorem 1 since $\mu_{q,1}(T: C(K) \to E) = \pi_{q'}(T: C(K) \to E)$. In the case $1 < p < q < 2$ we have with a constant $C_{p,q} > 0$,

$$\mu_{q,p}(I_n : l_{q'}^n \to l_{q'}^n) \geq C_{p,q}(1 + \log n)^{1/q} \quad \text{and} \quad \pi_{r,p}(I_n : l_{q'}^n \to l_{q'}^n) = 1,$$

$1/r + 1/q = 1/p$ (see [Pi]). Yet we do not know whether the factor $(1 + \log n)^{1/q}$ is asymptotically correct in the remaining cases $1 < p < 2 \leq q < \infty$ and $2 \leq p < q < \infty$, respectively.

3.3. Tensor products

Investigating tensor product operators between L_p-spaces, Rosenthal and Szarek [R–S] proved that for $1 \leq p < q < 2$,

$$t_n(p,q) := \sup_{\substack{\|S: l_q^n \to l_p^n\| \leq 1 \\ \|T: l_q^n \to l_p^n\| \leq 1}} \|S \otimes T : l_q^n \otimes_q l_q^n \to l_p^n \otimes_p l_p^n\| \geq C_{p,q}(1 + \log n)^{1/q},$$

where $C_{p,q} > 0$ is a constant depending on p and q only. Here the norm on the tensor product $l_q^n \otimes_q l_q^n$ is the norm induced from $l_q^{n^2}$. They conjectured that

$$t_n(p,q) \asymp (1 + \log n)^{1/q},$$

which will be answered in the positive. Our proof is based on the following inequality taken from [D–F] (which is a slight modification of a result in [C–D]).

Lemma C. *Let $1 \leq p, q \leq \infty$ and let $S, T \in \mathcal{L}(l_q^n, l_p^n)$. Then*

$$\|S \otimes T : l_q^n \otimes_q l_q^n \to l_p^n \otimes_p l_p^n\| \leq \mu_{s,p}(S')\mu_{s',q'}(T)$$

for all $1 \leq s \leq \infty$.

Proposition 4. *Let $1 \leq p < q < 2$. Then*

$$t_n(p,q) \asymp (1 + \log n)^{1/q}.$$

Proof. It remains to show the estimate from above. By Proposition 3 we have

$$\mu_{q,p}(I_n : l_{q'}^n \to l_{q'}^n) \leq C(1 + \log n)^{1/q}\pi_{r,p}(I_n : l_{q'}^n \to l_{q'}^n)$$

for $1/r + 1/q = 1/p$. Moreover, $\pi_{r,p} \leq \pi_{q',1}$ and

$$\pi_{q',1}(I_n : l_{q'}^n \to l_{q'}^n) = 1$$

(see [K], [Pi] or [T]) imply

$$\mu_{q,p}(I_n : l_{q'}^n \to l_{q'}^n) \leq C(1 + \log n)^{1/q}.$$

Choosing $s = q$ in Lemma C we obtain

$$\begin{aligned} \|S \otimes T: l_q^n \otimes_q l_q^n \to l_p^n \otimes_p l_p^n\| &\le \mu_{q,p}(S')\mu_{q',q'}(T) \\ &= \|T\|\mu_{q,p}(S') \le \|T\|\,\|S'\|\mu_{q,p}(I_n: l_{q'}^n \to l_{q'}^n) \\ &\le C(1 + \log n)^{1/q}\|S\|\,\|T\|. \end{aligned}$$

Hence

$$t_n(p,q) \le C(1 + \log n)^{1/q},$$

the desired estimate from above. ∎

3.4. Final remarks

In the study of (p,q)-summing norms of finite rank operators the following refinement of the definition is useful. Let $1 \le q \le p < \infty$ and let n be a positive integer, $\pi_{p,q}^{(n)}(T)$ of an operator $T \in \mathcal{L}(E,F)$ is the smallest $C \ge 0$ such that for arbitrary n-vectors $x_1, \ldots, x_n \in E$ one has

$$\left(\sum_{j=1}^{n} \|Tx_j\|^p\right)^{1/p} \le C \sup_{\|a\| \le 1} \left(\sum_{j=1}^{n} |\langle x_j, a\rangle|^q\right)^{1/q}.$$

The following inequality,

$$\pi_{p,2}(T) \le \sqrt{2}\pi_{p,2}^{(n)}(T),$$

is due to Tomczak-Jaegermann for $p = 2$ and has been extended by König to $p > 2$ with a constant C_p depending on p on the right-hand side of this inequality. Szarek showed that the constant C_p is independent of p. However, the final constant $\sqrt{2}$ has recently been obtained by Defant/Junge [D–J] using a clever characterization of (p,q)-summing operators as a quotient formula. Applying this inequality we get from Theorem 1, according to the pattern of Tomczak-Jaegermann [T], that

$$(*) \qquad \pi_p(T) \le C\left(\frac{1}{2} - \frac{1}{p}\right)^{-1/p'} (1 + \log n)^{1-1/p}\pi_{p,1}^{(n)}(T), \qquad p > 2,$$

for rank n operators from a $C(K)$-space. In the limit case $p = 2$ we even get an improvement of Jameson's result

$$\pi_2(T) \le C(1 + \log n)^{1/2}\pi_{2,1}^{(n)}(T)$$

which is a consequence of Tomczak-Jaegermann's inequalities

$$\pi_2(T) \le \sqrt{2}\pi_2^{(n)}(T)$$

and

$$\pi_2^{(n)}(T) \le C(1+\log n)^{1/2}\pi_{2,1}^{(n)}(T),$$

hence it is due to Tomczak-Jaegermann [T]. Finally, we should mention that in the case $2 < p < \infty$ we may also give an alternative proof of the inequality (*) using Maurey's result and the recent result of Defant/Junge [D–J] instead of Lemma B. Namely, for rank n operators from $C(K)$, the inequality (*) follows by combining the inequalities

$\pi_p(T) \le \frac{a}{p}\left(\frac{1}{r}-\frac{1}{p}\right)^{-1/r'}\pi_{r,2}(T),$	$2 \le r < p < \infty$	(Maurey),
$\pi_{r,2}(T) \le \sqrt{2}\pi_{r,2}^{(n)}(T),$	$2 \le r \le \infty$	(Defant/Junge),
$\pi_{r,2}^{(n)}(T) \le n^{1/r-1/p}\pi_{p,2}^{(n)}(T),$	$2 \le r \le p < \infty$	(Hölder's inequality),
$\pi_{p,2}^{(n)}(T) \le \frac{C}{2}\left(\frac{1}{2}-\frac{1}{p}\right)^{-1/p'}\pi_{p,1}^{(n)}(T),$	$2 < p < \infty$	(Maurey, Tomczak-Jaegermann),

and by choosing again an appropriate r, $1/r = 1/p + (1/2 - 1/p)(1+\log n)^{-1}$, similar as in the proof of the main theorem. However, we do not know whether the inequality (*) remains valid for $1 < p < 2$.

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